

ON L^p ESTIMATES FOR INTEGRAL TRANSFORMS

BY
T. WALSH

Abstract. In a recent paper R. S. Strichartz has extended and simplified the proofs of a few well-known results about integral operators with positive kernels and singular integral operators. The present paper extends some of his results. An inequality of Kantorovič for integral operators with positive kernel is extended to kernels satisfying two mixed weak L^p estimates. The "method of rotation" of Calderón and Zygmund is applied to singular integral operators with Banach space valued kernels. Another short proof of the fractional integration theorem in weighted norms is given. It is proved that certain sufficient conditions on the exponents of the L^p spaces and weight functions involved are necessary. It is shown that the integrability conditions on the kernel required for boundedness of singular integral operators in weighted L^p spaces can be weakened. Some implications for integral operators in R^n of Young's inequality for convolutions on the multiplicative group of positive real numbers are considered. Throughout special attention is given to restricted weak type estimates at the endpoints of the permissible intervals for the exponents.

0. Introduction. The present paper attempts to answer a few questions raised by a paper of Strichartz [25] and the following example may be illustrative of its contents. As usual, for $x \in R^n$ let $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. It was shown by Stein [22] that if Ω is positively homogeneous of degree zero, bounded and of mean value zero on the unit sphere S^{n-1} in R^n and $K(x) = \Omega(x)|x|^{-n}$ then the singular integral operator T defined by $Tf(x) = \text{p.v. } K * f(x) = \lim_{\epsilon \rightarrow 0} \int_{|x-t| > \epsilon} K(x-t)f(t) dt$ satisfies

$$\int |Tf(x)|^p |x|^{\alpha p} dx \leq A_p^p \int |f(x)|^p |x|^{\alpha p} dx$$

for $1 < p < \infty$, $-n/p < \alpha < n/p'$. Here and in what follows p' denotes the exponent conjugate to p , i.e., $1/p + 1/p' = 1$. Strichartz showed that it is sufficient to require $\Omega \in L^q(S^{n-1})$, $q = \max(p, p')$, $q > 2(n-1)$ and conjectured that the condition $q > 2(n-1)$ is not necessary (Theorem 6 of [25]). It is proved in the present paper that $1/q = 1 - |\alpha|/n$ is sufficient which in particular shows that $q > 2(n-1)$ is not needed (Proposition 7).

NOTATION. As in [25] let (X, Σ, μ) , (Y, Σ', ν) be totally σ -finite measure spaces, $K(x, y)$ a measurable function on $X \times Y$ and define $Tf(x) = \int K(x, y)f(y) dy$ where

Received by the editors January 9, 1970 and, in revised form, May 29, 1970.

AMS 1969 subject classifications. Primary 4425, 4428, 4430, 4432, 4450.

Key words and phrases. L^p estimates, integral transforms, inequality of Kantorovič, complex interpolation, Lorentz spaces, singular integrals, fractional integration, weighted L^p spaces, Schur's inequality.

Copyright © 1971, American Mathematical Society

for simplicity dy, dx are written for $d\nu(y), d\mu(x)$. For f measurable let λ_f, f^* denote the distribution function and decreasing rearrangement of f on $(0, \infty)$ respectively. Further as in [13] set

$$\|f\|_{pq}^* = \|f\|^*[L^{pq}(X)] = \left(q/p \int_0^\infty [f^*(t)t^{1/p}]^q dt/t \right)^{1/q},$$

$$\|f\|_{pq} = \|f\|[L^{pq}(X)] = \|f^{**}\|_{pq}^*$$

where

$$f^{**}(t) = f^{**}(t, r) = \sup \left\{ \left(|E|^{-1} \int |f(x)|^r dx \right)^{1/r} : |E| \geq t \right\},$$

$$0 < r \leq 1, \quad r \leq q, \quad r < p$$

and for simplicity $|E| = \mu(E)$. If F is a Banach (an s -Banach) space $L^p(X, F)$ will denote the space of F -valued measurable functions on X such that $\|f\|_F \in L^p(X)$. Also let $L^{(p,q)}(X \times Y) = L^q(Y, L^p(X))$. The norm of $L^{(p,q)}$ will also be denoted $\|\cdot\|_{(p,q)}$ (cf. [2]). For f measurable on $X \times Y$ define

$$X^{p_1 q_1} Y^{p_0 q_0} f = \|\varphi\|_{p_1 q_1} \quad \text{where } \varphi(x) = \|f(x, \cdot)\|_{p_0 q_0}.$$

It will be convenient to define $X_*^{p_1 q_1} Y_*^{p_0 q_0} f$ analogously by replacing $\|\cdot\|_{p_i q_i}$ by the quasi-norm $\|\cdot\|_{p_i q_i}^*$ for $i=0, 1$.

1. An extension of an inequality of Kantorovič. It is well known (see [14]) that if

$$(1) \quad X^{r_0} Y^{s_0} K = M_0 < \infty, \quad Y^{s_1} X^{r_1} K = M_1 < \infty$$

then

$$(2) \quad \|Tf\|_r \leq M_0^{1-\theta} M_1^\theta \|f\|_s,$$

provided, for some θ , such that $0 \leq \theta \leq 1$

$$(3) \quad 1/r = (1-\theta)/r_0 + \theta/r_1, \quad 1/s = (1-\theta)/s_0 + \theta/s_1,$$

$$(4) \quad (1-\theta)/s_0 + \theta/r_1 \leq 1, \quad (s \geq 1).$$

If the hypotheses are weakened by replacing the L^p (quasi-) norms in (1) by weak L^p (i.e., $L^{p\infty}$) (quasi-) norms there is a corresponding weaker conclusion given by

PROPOSITION 1. Suppose $X^{r_0\infty} Y^{s_0\infty} K = M_0$, $Y^{s_1\infty} X^{r_1\infty} K = M_1$ and (3) is satisfied with $0 \leq \theta \leq 1$. Then

$$\|Tf\|_{s,\infty} \leq B M_0^{1-\theta} M_1^\theta \|f\|_{s,1}$$

provided $s > 1$ and

$$(5) \quad (1-\theta)/s_0 + \theta/r_1 < 1.$$

Here B denotes a finite constant depending on r_0, r_1, s_0, s_1 , and θ . If also $0 < \theta < 1$, $r_0 \neq r_1$, $0 < u \leq \infty$ (and $M_0, M_1 \geq 1$) then

$$\|Tf\|_{ru} \leq B(M_0 + M_1)\|f\|_{s'u}.$$

This is Theorem 2 of [25] except that the assumptions on the exponents are less restrictive. Theorem 2 of [25] in fact follows from the Marcinkiewicz interpolation theorem for $L^{p,q}$ spaces (see [13]) and the following

LEMMA 1. Suppose $M_0 = X^{qt} Y^{p's'} K < \infty$,

$$(6) \quad 1 < p, 1 \leq s \quad \text{or} \quad p = s = 1 \quad \text{and} \quad q, t > 0$$

then

$$(7) \quad \|Tf\|_{qt} \leq BM_0 \|f\|_{ps}.$$

If $M_1 = Y^{p's'} X^{qt} K < \infty$,

$$(8) \quad 1 < p, 1 \leq s \quad \text{or} \quad p = s = 1 \quad \text{and} \quad 1 < q, 1 \leq t \quad \text{or} \quad q = t = 1$$

then

$$(9) \quad \|Tf\|_{qt} \leq BM_1 \|f\|_{ps}.$$

Proof. Let $K, f \geq 0$. (7) follows from $\int K(x, y)f(y) dy \leq B\|K(x, \cdot)\|_{p's'}\|f\|_{ps}$. As for (9), by Minkowski's inequality for integrals, provided L^{qt} is a normed space

$$\left\| \int K(\cdot, y)f(y) dy \right\|_{qt} \leq \int \|K(\cdot, y)\|_{qt} f(y) dy \leq BM_1 \|f\|_{ps}.$$

With the present notation which differs somewhat from that of [25], if $1 < s_0 < \infty$, $1 < r_1 < \infty$, $0 < r_0 \leq \infty$, $1 < s_1 \leq \infty$ then by Lemma 1

$$\|Tf\|_{r_0\infty} \leq BM_0 \|f\|_{s_01} \quad \text{and} \quad \|Tf\|_{r_1\infty} \leq BM_1 \|f\|_{s_11}$$

so that the Marcinkiewicz interpolation theorem implies Proposition 1 in this case.

Proof of Proposition 1. Consider $T(K_0, K_1, f)(x) = \int K_0(x, y)K_1(x, y)f(y) dy$. Suppose $X^{r_0u_0}Y^{s_0v_0}K_0 < \infty$, $Y^{s_1v_1}X^{r_1u_1}K_1 < \infty$ and furthermore that there exists t such that $0 \leq t \leq 1$ and $1 - 1/(s_0(1-t))$, $1 - 1/(v_0(1-t))$ in place of $1/p$, $1/s$ satisfy (6) and $1 - 1/(s_1t)$, $1 - 1/(v_1t)$, $1/(r_1t)$, $1/(u_1t)$ satisfy (8) in place of $1/p$, $1/s$, $1/q$, $1/t$. Then by Lemma 1

$$\begin{aligned} \|T(K_0, K_1, f)\|_{r_0(1-t), u_0(1-t)} \\ \leq BX^{r_0(1-t), u_0(1-t)} Y^{s_0(1-t), v_0(1-t)}(K_0) \|K_1\|_{\infty} \|f\|_{[s_0(1-t)]', [v_0(1-t)]'} \end{aligned}$$

and

$$\|T(K_0, K_1, f)\|_{r_1t, u_1t} \leq B\|K_0\|_{\infty} Y^{s_1t, v_1t} X^{r_1t, u_1t}(K_1) \|f\|_{(s_1t)', (v_1t)'},$$

hence by the complex method of interpolation (see 10.1 of [5] which extends to s -Banach spaces, see [9], [13], required since $L^{r_0 u_0}$ is not necessarily a Banach space) with a possibly different B

$$(10) \quad \|T(K_0, K_1, f)\|_{ru} \leq BX^{r_0 u_0} Y^{s_0 v_0}(K_0) Y^{s_1 v_1} X^{r_1 u_1}(K_1) \|f\|_{s'v'}$$

where $r^{-1} = r_0^{-1} + r_1^{-1}$, $u^{-1} = u_0^{-1} + u_1^{-1}$, $s^{-1} = s_0^{-1} + s_1^{-1}$, $v^{-1} = v_0^{-1} + v_1^{-1}$. By the above, t must satisfy $\max(r_1^{-1}, s_1^{-1}) \leq t \leq 1 - s_0^{-1}$ hence $s \geq 1$ and

$$(11) \quad 1/s_0 + 1/r_1 \leq 1.$$

If on the other hand the latter two inequalities hold $0 \leq t \leq 1$ and the inequalities for r_i, s_i can be satisfied. Thus if $r_i = u_i$, $s_i = v_i$, (11) (and $s \geq 1$) is sufficient for (10). (In this case $B = 1$.)

If $u_i = v_i = \infty$ for $i = 0, 1$, Lemma 1 requires $\max(1/r_1, 1/s_1) < t < 1 - 1/s_0$ hence $s > 1$, $1/s_0 + 1/r_1 < 1$ is sufficient for

$$\|T(K_0, K_1, f)\|_{r\infty} \leq BX^{r_0 \infty} Y^{s_0 \infty}(K_0) Y^{s_1 \infty} X^{r_1 \infty}(K_1) \|f\|_{s1}.$$

Inequalities of Kantorovič's type can be deduced from this by letting

$$(12) \quad K_0 = K^{1-\theta}, \quad K_1 = K^\theta \quad \text{for some } 0 \leq \theta \leq 1.$$

In fact it follows from the definition of $\|\cdot\|_{pq}$ that $\| |f|^\theta \|_{pq} = \|f\|_{p\theta, q\theta}^\theta$ for $\theta > 0$. (If $f^{**}(t, r)$ is used in the definition of $\|f\|_{pq}$ then r/θ has to take the place of r in the definition of $\| |f|^\theta \|_{pq}$.) From the hypotheses of Proposition 1

$$\begin{aligned} X^{r_0/(1-\theta), u_0/(1-\theta)} Y^{s_0/(1-\theta), v_0/(1-\theta)} (|K|^{1-\theta}) &= M_0^{1-\theta}, \\ Y^{s_1/\theta, v_1/\theta} X^{r_1/\theta, u_1/\theta} (|K|^\theta) &= M_1^\theta. \end{aligned}$$

Hence if (3) is satisfied and $r_i = u_i$, $s_i = v_i$ for $i = 0, 1$, then $\|Tf\|_r \leq M_0^{1-\theta} M_1^\theta \|f\|_{s'}$ provided $(1-\theta)/s_0 + \theta/r_1 \leq 1$ and $(s \geq 1)$ which is Kantorovič's inequality (2). The proposition follows likewise from the above.

That in general the condition $(1-\theta)/s_0 + \theta/r_1 < 1$ cannot be weakened beyond replacement of $<$ by \leq can be shown as follows.

PROPOSITION 2. *Let X, Y both have the property that for any two measurable subsets E_1, E_2 , $E_1 \subset E_2$ of finite measure there is another subset E'_1 disjoint from E_2 such that the measures of E_1 and E'_1 are the same. Then in order for*

$$T(K_0, K_1, f) = \int K_0(\cdot, y) K_1(\cdot, y) f(y) dy$$

to satisfy

$$(13) \quad \|T(K_0, K_1, f)\|_{q\infty} \leq BX^{r_0 u_0} Y^{s_0 v_0}(K_0) Y^{s_1 v_1} X^{r_1 u_1}(K_1) \|f\|_{pw}$$

(with B independent of K_0, K_1, f), it is necessary that

$$(14) \quad 1/r_0 + 1/s_1 + 1/p - 1/q \geq 0.$$

Proof. It seems slightly more convenient to work with the quasi-norms $\|\cdot\|^*$. Let then B be minimal in (13) with the norms starred and all of K_0, K_1, f vanishing outside sets of finite measure, of the special form $E \times F$ in the case of K_0, K_1 . For $0 < \eta < 1$ there exist then simple functions K_0, K_1, f such that

$$(15) \quad K_i = \sum_{k,l=1}^{k_1, l_1} a_i^{kl} \chi_k \varphi_l, \quad f = \sum_{l=1}^{l_1} b_l \varphi_l, \quad a_i^{kl}, b_l \in R, \quad i = 0, 1$$

where χ_k, φ_l are the characteristic functions of two sequences of mutually disjoint sets $\{E_k\}_{1 \leq k \leq k_1}$ and $\{F_l\}_{1 \leq l \leq l_1}$ respectively, and for which

$$(16) \quad \|T(K_0, K_1, f)\|_{q\infty}^* > B\eta \|K_0\|^* \|K_1\|^* \|f\|_{pw}^* \quad (\|K_0\|^* = X_*^{r_0 u_0} Y_*^{s_0 v_0}(K_0)).$$

By hypothesis there exist sequences of mutually disjoint sets $\{E'_k\}_{1 \leq k \leq k_1}$, $\{F'_l\}_{1 \leq l \leq l_1}$ in X, Y contained in the complement of $\bigcup_{k=1}^{k_1} E_k$ and $\bigcup_{l=1}^{l_1} F_l$ respectively such that $|E_k| = |E'_k|$, $|F_l| = |F'_l|$ for $1 \leq k \leq k_1$, $1 \leq l \leq l_1$. Let χ'_k, φ'_l be the characteristic functions of E'_k, F'_l . Define

$$\begin{aligned} K'_i &= \sum a_i^{kl} \chi'_k \varphi'_l, & f' &= \sum_{l=1}^{l_1} b_l \varphi'_l, \\ K_i^{\sim} &= K_i + K'_i, & f^{\sim} &= f + f', \quad i = 0, 1, \end{aligned}$$

then $\lambda_{f^{\sim}}(t) = 2\lambda_f(t)$ hence $(f^{\sim})^*(t) = f^*(t/2)$. Thus

$$(17) \quad \|f^{\sim}\|_{pw}^* = 2^{1/p} \|f\|_{pw}^*.$$

Similarly

$$\|K_0^{\sim}\|^* = 2^{1/r_0} \|K_0\|^*, \quad \|K_1^{\sim}\|^* = 2^{1/s_1} \|K_1\|^*,$$

$$\|T(K_0^{\sim}, K_1^{\sim}, f^{\sim})\|_{q\infty}^* = \|T(K_0, K_1, f) + T(K'_0, K'_1, f')\|_{q\infty}^* = 2^{1/q} \|T(K_0, K_1, f)\|_{q\infty}^*.$$

So by the definition of B

$$2^{1/q} \|T(K_0, K_1, f)\|_{q\infty}^* \leq B 2^{1/r_0 + 1/s_1 + 1/p} \|K_0\|^* \|K_1\|^* \|f\|_{pw}^*.$$

This and (16) imply $2^{1/r_0 + 1/s_1 + 1/p - 1/q} > \eta$. Since $\eta < 1$ was arbitrary (14) follows.

REMARK. Proposition 2 holds in particular if X, Y are noncompact locally compact (second countable) groups provided with left-invariant measure. The above argument was suggested by the proof of Theorem 1.1 of [12]. If $u < \infty$ the simple functions are dense in L^{pu} . Hence it follows from (17) that likewise if T is a continuous linear operator from $L^{pu}(G)$ to $L^{qv}(G)$ where G is a noncompact locally compact group, $T(f(a.)) = (Tf)(a.)$ for any $a \in G$ and $q < p < \infty$, $v < \infty$, then $T = 0$ (u may be replaced by any smaller positive exponent). In case $v = \infty$, $1 < q < p < \infty$, $u > 1$, it follows by consideration of the adjoint of T that $T = 0$.

LEMMA 2. Let X, Y be, more particularly, continuous measure spaces of infinite measure such that (13) is valid for all K_0, K_1, f then

$$(18) \quad 1/q = 1/r_0 + 1/r_1, \quad 1/p + 1/s_0 + 1/s_1 = 1.$$

Proof. For any triple K_0, K_1, f of simple functions of the special form (15) and any $\varepsilon, \eta > 0$ there are mutually disjoint sets E_k^ε and F_l^η such that $|E_k^\varepsilon| = \varepsilon^{-1}|E_k|$, $|F_l^\eta| = \eta^{-1}|F_l|$. Let $\chi_k^\varepsilon, \varphi_l^\eta$ be the corresponding characteristic functions

$$f^\eta = \sum_{i=1}^{l_1} b_i \varphi_i^\eta, \quad K_i^{\varepsilon, \eta} = \sum a_i^{k,l} \chi_k^\varepsilon \varphi_l^\eta, \quad i = 0, 1.$$

Then

$$\|f^\eta\|_{pw}^* = \eta^{-1/p} \|f\|_{pw}^*, \quad \|K_i^{\varepsilon, \eta}\|^* = \varepsilon^{-1/r_i} \eta^{-1/s_i} \|K_i\|^*, \quad i = 0, 1,$$

and

$$\|T(K_0^{\varepsilon, \eta}, K_1^{\varepsilon, \eta}, f^\eta)\|_{q\infty}^* = \eta^{-1} \varepsilon^{-1/q} \|T(K_0, K_1, f)\|_{q\infty}^*.$$

Hence $\varepsilon^{-1/q+1/r_0+1/r_1} \eta^{-1+1/s_0+1/s_1+1/p}$ is bounded for $\varepsilon, \eta > 0$ and (18) follows. Note that in the present situation (14) is equivalent to (11).

COROLLARY. If for $T(K, f)(x) = \int K(x, y)f(y) dy$,

$$\|T(K, f)\|_{q\infty} \leq BX^{r_0 u_0} Y^{s_0 v_0} (K)^{1-\theta} Y^{s_1 v_1} X^{r_1 u_1} (K)^\theta \|f\|_{pw}$$

with B independent of K, f , then, in the situation of the preceding lemma, (3) and (4) must be satisfied.

This follows similarly by proceeding with K as with K_0, K_1 above.

2. Variable kernel singular integral operators. §4 of [25] is concerned with singular integrals of the form

$$(19) \quad Tf(x) = \text{p.v.} \int H(x-y)f(y) dy(x)$$

where $H(t, x) = H(x)(t) = \Omega(t, x')|x|^{-n}$ for $x' = |x|^{-1}x$ and Ω is a function on S^{n-1} with values in the space L_u^2 (L_u^2 consists of the tempered distributions f on R^n such that $\hat{f}(1+|\cdot|)^{u/2} = \hat{g}$, where $g \in L^q$, in this case $\|f\|_{q,u} = \|g\|_q$, see [4]). By heavier use of the results of Calderón and Zygmund about singular integrals based on the "method of rotation" in [8] stronger results can be obtained for L_u^q valued kernels. In particular Ω need not satisfy any smoothness condition (as a function on S^{n-1}).

PROPOSITION 3. Let Z be some measure space, $\Omega \in L^{(q,1)}(R^n \times S^{n-1})$, $1 < q < \infty$. Suppose also the even part Ω_0 , say, of Ω is in $L \log^+ L(S^{n-1}, L^q(Z))$ and has mean value zero (in $L^q(Z)$) on S^{n-1} . Let the completed tensor product of $L^p(R^n)$ and $L^q(Z)$ with the projective norm (see [26]) be denoted $L^p \tilde{\otimes} L^q = L^p(R^n) \tilde{\otimes} L^q(Z)$. Then, if also $\Omega_1 = \Omega - \Omega_0$, $1 < p < \infty$,

$$(20) \quad \|\text{p.v. } H * f\| [L^p \tilde{\otimes} L^q] \leq C_{p,q} (1 + \|\Omega_0\| [L \log^+ L(S^{n-1}, L^q(Z))] + \|\Omega_1\| [L(S^{n-1}, L^q(Z))]) \|f\|_p.$$

Proof. Suppose first of all Ω is odd, i.e., $\Omega(-x) = -\Omega(x)$. Then (see [8])

$$\int_{|x-y|>\varepsilon} H(\cdot, x-y)f(y) dy = \int_{S^{n-1}} \Omega(\cdot, y') \tilde{f}_\varepsilon(x, y') dy'$$

where $\tilde{f}_\varepsilon(x, y') = \int_{|t| > \varepsilon} f(x - ty') t^{-1} dt$ and dy' denotes the element of area on S^{n-1} . Hence

$$\begin{aligned} \left\| \int_{|\cdot - y| > \varepsilon} H(\cdot, \cdot - y) f(y) dy \right\| [L^q(Z) \tilde{\otimes} L^p(R^n)] \\ \leq \int_{S^{n-1}} \|\Omega(\cdot, y') \tilde{f}_\varepsilon(\cdot, y')\| [L^q(Z) \otimes L^p(R^n)] dy' \\ = \int_{S^{n-1}} \|\Omega(\cdot, y')\|_q \|\tilde{f}_\varepsilon(\cdot, y')\|_p dy' \leq C_p \|\Omega\|_{(q,1)} \|f\|_p. \end{aligned}$$

It remains to consider the case of even Ω . It is sufficient to observe that the reduction to the case of odd kernels via the formula $f = \sum_{i=1}^n R_i(R_i f)$, where $Rf = (R_1 f, \dots, R_n f)$ denotes the Riesz transform of f , given in [8], can be carried over to kernels with values in $L^q(Z)$ for $1 < q < \infty$. This argument is based on the fact that the Riesz transform of a real valued function in $L(\log^+ L + 1)(R^n)$ is locally integrable. The analogous result holds for L^q valued functions. For if K denotes the Riesz kernel $K(x) = c_n^{-1} |x|^{-n-1} x$ then for all $z \in Z$

$$\|K_\varepsilon * f(z, \cdot)\|_q^q \leq C_q^q \|f(z, \cdot)\|_q^q$$

and integration over Z gives

$$\|K_\varepsilon * f\| [L^q(R^n, L^q(Z))] \leq C_q \|f\| [L^q(R^n, L^q(Z))].$$

Also K satisfies the condition

$$(21) \quad \int_{|x| \geq 2|y|} |K(x-y) - K(x)| dx \leq C \quad \text{for all } y \in R^n.$$

Hence by Theorem 2 of [1]

$$\|Rf\| [L^p(R^n, L^q(Z))] \leq \lim_{\varepsilon \rightarrow 0} \|K_\varepsilon * f\| \leq C_{p,q} \|f\| [L^p(R^n, L^q(Z))]$$

where $C_{p,q} = O((p-1)^{-1})$ for $p \downarrow 1$. According to [27, p. 119, 4.41(ii)] the latter estimate implies that for $f \in L(\log^+ L + 1)(R^n, L^q(Z))$, $Rf \in L_{\text{loc}}(R^n, L^q(Z))$.

An extension of the argument of [8] to L^q valued kernels to prove (20) also requires an estimate analogous to (20) for an L^q valued maximal function defined by

$$T^* f(x) = \sup_{\varepsilon > 0} \varepsilon^{-n} \left| \int_{|y| \leq \varepsilon} \Omega^*(\cdot, y') f(x-y) dy \right| \quad \text{where } \Omega^* \in L^1(S^{n-1}, L^q(Z)).$$

That in fact

$$\|T^* f\| [L^p \tilde{\otimes} L^q] \leq C_p \|\Omega\| [L(S^{n-1}, L^q(Z))] \|f\|_p$$

for $1 < p \leq \infty$, $1 \leq q \leq \infty$ follows by analogous modification of the argument of [8].

By application of continuous linear operators corresponding to continuous bilinear operators on $L^p \times L^q$ to the left-hand side of (20) various corollaries can be obtained.

COROLLARY 1. Suppose the measurable function $\Omega \in L^1(S^{n-1}, L_u^b(R^n))$, where $u > q/n$, Ω has mean value zero on S^{n-1} and the L_u^q -norm of the even part of Ω belongs to $L \log^+ L(S^{n-1})$. Then T defined by (19) satisfies $\|Tf\|_p \leq C_{p,q,u}(\Omega)\|f\|_p$ for $1 < p < \infty$.

Proof. Let J^u denote the isomorphism between L^q and L_u^q defined by $(J^u f)^\wedge = (1 + |\cdot|^2)^{-u/2} \hat{f}$ and define Ω^{-u} by $\Omega^{-u}(\cdot, x) = J^{-u} \Omega(\cdot, x)$ and H^{-u} analogously. Then by Proposition 3

$$\|p.v. H^{-u} * f\| [L^p \tilde{\otimes} L^q] \leq C_p(\Omega) \|f\|_p.$$

Hence

$$\|p.v. H * f\| [L^p \tilde{\otimes} L_u^q] \leq C_p(\Omega) \|f\|_p.$$

Now by Sobolev's inequality for any $x \in R^n$,

$$\|p.v. H * f(\cdot, x)\|_\infty \leq \|p.v. H * f(\cdot, x)\|_{p,u},$$

hence

$$\begin{aligned} \|Tf\|_p &\leq C_{q,u} \|p.v. H * f\| [L^p(R^n), L_u^q(R^n)] \\ &\leq C_{q,u} \|p.v. H * f\| [L^p \tilde{\otimes} L_u^q] \leq C_{p,q,u}(\Omega) \|f\|_p. \end{aligned}$$

REMARK. Suppose $\Omega \in L^1(S^{n-1}, L^p(R^n))$ and now $f \in L_u^q$ while the remaining conditions are altered in the obvious manner. Let $Tf(x)$ be defined by evaluation of the (continuous) function $(p.v. H * f)(x, \cdot) = p.v. H * f(\cdot)(x) \in L_u^q$ at x . Then $\|Tf\|_p \leq C_{p,q,u}(\Omega) \|f\|_{q,u}$. This follows similarly from $\|\hat{f}_\varepsilon(\cdot, y')\|_{q,u} \leq C_{q,u} \|f\|_{q,u}$ (a consequence of the fact that the operation $f \rightarrow \hat{f}_\varepsilon(\cdot, y')$ commutes with J^u).

For simplicity the next corollary is stated only for odd kernels.

COROLLARY 2. Suppose $\Omega(x, y') = -\Omega(x, -y')$ and $\Omega \in L^{(q,1)}(R^n \times S^{n-1})$, $1 < p < \infty$, $1 \geq 1/r = 1/p + 1/q$ then T defined by (19) satisfies

$$(22) \quad \|Tf\|_r \leq C_p \|\Omega\|_{(q,1)} \|f\|_p.$$

Proof. By Proposition 3, (20) is satisfied ($Z = R^n$). By Hölder's inequality the bilinear map $(f, g) \rightarrow fg$ is continuous from $L^p(R^n) \times L^q(R^n)$ to L^r . This implies (22).

Corollary 2 might be compared with Theorem 4.1 of [7] which deals with kernels in $L^{(s,q)}(S^{n-1} \times R^n)$ where $1 < s$.

3. Fractional integration in weighted norms. First a result needed repeatedly later will be proved.

LEMMA 3. Let f be measurable on $X \times Y$ then

$$(23) \quad \|f\|_{p\infty}^* \leq X^p Y_*^{p\infty} f.$$

(For the case $f(x, y) = f_1(x)f_2(y)$ see O'Neil [19, Theorem 11.4].)

Proof.

$$\begin{aligned}\|f\|_{p\infty}^{*p} &= \sup_{t>0} t^p \lambda_f(t) = \sup_{t>0} t^p \int_X \lambda_{f(x,\cdot)}(t) dx \leq \int_X \sup_{t>0} t^p \lambda_{f(x,\cdot)}(t) dx \\ &= \int_X \|f(x, \cdot)\|_{p\infty}^{*p} dx.\end{aligned}$$

REMARKS. Obviously, if for a.e. x and all $t > 0$, $t^p \lambda_{f(x,\cdot)}(t) = \|f(x, \cdot)\|_{p\infty}^{*p}$ then equality holds in (23). Thus if $f = f_1 f_2$, $f_1 \in L^p(X)$, f_2 measurable on Y and such that $\lambda_{f_2}(t) = C t^{-p}$ then $t^p \lambda_{f(x,\cdot)}(t) = t^p \lambda_{f_2}(t/|f_1(x)|) = C |f_1(x)|^p$ and hence $\|f\|_{p\infty}^{*p} = C \|f\|_p^p$ (cf. [19]). Since $\|\cdot\|_{p\infty}^*$ and $\|\cdot\|_{p\infty}$ are equivalent it is also true that $\|f\|_{p\infty} \leq C_p X^p Y^{p\infty} f$. In fact $C_p = 1$. For if $r \leq 1$, $r < p$ and $E_x = \{y : (x, y) \in E\}$; then

$$\begin{aligned}\|f\|_{p\infty} &= \sup_{t>0} t^{1/p} f^{**}(t, r) = \sup_{|E| \geq t} t^{1/p} \left(|E|^{-1} \int_E |f(x, y)|^r dx dy \right)^{1/r} \\ &\leq \sup_{|E| \geq t} t^{1/p} \left(|E|^{-1} \int_X |E_x|^{-r/p+1} \|f(x, \cdot)\|_{p\infty}^r dx \right)^{1/r} \\ &\leq \sup_{|E| \geq t} t^{1/p} |E|^{-1/r} \left(\int_X |E_x| dx \right)^{1/r-1/p} X^p Y^{p\infty} f.\end{aligned}$$

For $p < \infty$ there exist functions f such that $\|f\|_{p\infty} = \infty$ while $X^{p\infty} Y^p f < \infty$. It is sufficient to show this for $p = 1$. Let $X = Y = R_+$ and f the characteristic function of $\{(x, y) : xy < 1\}$ then $\lambda_f = \infty$ while $X_*^{1\infty} Y^1 f = \|\cdot\|_1^* = 1$.

COROLLARY. For f measurable on $X \times Y$, $1 \leq p < \infty$, $X^p Y_*^{p1} f \leq \|f\|_{p1}^*$.

Proof.

$$\begin{aligned}X^p Y_*^{p1} f &= \sup_{\|g\|_{p'} \leq 1} \int \|f(x, \cdot)\|_{p1}^* g(x) dx \\ &= p^{-1} \sup_{X^p Y_*^{p\infty} g \leq 1} \int f(x, y) g(x, y) dx dy \\ &\leq p^{-1} \sup_{\|g\|_{p'}^* \leq 1} \int f(x, y) g(x, y) dx dy \\ &= \|f\|_{p1}^*.\end{aligned}$$

By interpolation between these inequalities and $\|f\|_p = X^p Y^p f$ it follows that there is a $B_p > 0$ such that $\|f\|_{pu} \leq B_p X^p Y^{pu} f$ for $0 < p \leq u \leq \infty$ and $X^p Y^{pu} f \leq B_p \|f\|_{pu}$ for $1 \leq u \leq p < \infty$.

EXAMPLE. Consider $g(x) = \Omega(x)|x|^{-\lambda}$ where Ω is positively homogeneous of degree zero and $\|\Omega\|_{[L^{n/\lambda}(S^{n-1})]} < \infty$. Since $dx = dx' r^{n-1} dr$ for $x = rx'$, $|x'| = 1$, it follows that

$$\begin{aligned}\|r^{-\lambda}\|_{[L^{n/\lambda\infty}(R_+, r^{n-1} dr)]} &= \sup t^{\lambda/n} \int_{r^{-\lambda} > t} r^{n-1} dr \\ &= \sup t^{\lambda/n} \int_0^{t^{-1/\lambda}} r^{n-1} dr = n^{-1},\end{aligned}$$

hence $\|g\|_{n/\lambda, \infty}^* \leq n^{-1} \|\Omega\|_{n/\lambda}$. (By the above remark equality holds.) This implies Muckenhoupt's theorem [17]:

$$(24) \quad \|g * f\|_q \leq C \|f\|_p, \quad 1/q = 1/p - (n - \lambda)/n, \quad 0 < 1/q < 1/p < 1,$$

(cf. [19, 11.17]).

Another short proof of the fractional integration theorem in weighted norms can be given by means of the multiplication and convolution theorems for L^{pq} spaces with a somewhat enlarged range of values for p, q (see also [10]). As usual, let $\alpha^+ = \max(\alpha, 0)$, $\alpha^- = \max(-\alpha, 0)$.

PROPOSITION 4. Suppose $1/p' + 1/q = (\alpha + \beta + \gamma)/n$,

(25) $0 \leq \gamma < n$, $\alpha + \beta \geq 0$, $1/p' \geq \alpha^+/n$, $1/q \geq \beta^+/n$, $0 < u \leq v$ except if $1/p' = \alpha^+/n$ or $1/q = \beta^+/n$ (or both, hence $\alpha, \beta \geq 0$, $\gamma = 0$) in which case $1 = u = v'$ and $I_\gamma f(x) = \int |x - t|^{-\gamma} f(t) dt$, $\|f\|_{pu, \alpha} = \|f| \cdot |^\alpha\|_{pu}$. Then

$$\|I_\gamma f\|_{qv, -\beta} \leq C \|f\|_{pu, \alpha}.$$

Proof. Let $M_\alpha f(t) = |t|^{-\alpha} f(t)$, $I_\alpha f(t) = | \cdot |^{-\alpha} * f(t)$, $Tf(x) = \int K(x, y) f(y) dy$ where $K(x, y) = K_{\alpha, \gamma, \beta}(x, y) = |x|^{-\beta} |x - y|^{-\gamma} |y|^{-\alpha}$ and distinguish three cases.

(I) Suppose $\alpha, \beta \geq 0$. Then by the multiplication and convolution theorems (see [13, Theorems 4.5, 4.10]),

$$\|Tf\|_{qv} = \|M_\beta I_\gamma M_\alpha f\|_{qv} \leq C \|I_\gamma M_\alpha f\|_{p_2 v} \leq C \|M_\alpha f\|_{p_1 u} \leq C \|f\|_{pu}$$

where $1/q = 1/p_2 + \beta/n$, $1/p_2 = 1/p_1 + \gamma/n - 1$, $1/p_1 = 1/p + \alpha/n$, since by assumption $1 \geq 1/p_1 > 1/p_2 = 1/p + (\alpha + \gamma)/n - 1 = 1/q - \beta/n \geq 0$ with $1/p_1 = 1$ or $1/p_2 = 0$ only if $u = v' = 1$.

If one of $\alpha, \beta < 0$ estimate $K(x, y)$ separately according as (a) $|x|/|y| < 1/2$, (b) $1/2 \leq |x|/|y| \leq 2$, (c) $2 < |x|/|y|$.

(II) $\alpha < 0$, now $1/p' \geq 0$, $1/q \geq \beta/n > (\alpha + \beta)/n$. If (a) or (b) holds then $K(x, y) \leq C |x - y|^{-\gamma} |y|^{-\alpha - \beta}$, if (c) $K(x, y) \leq C |x - y|^{-\alpha - \gamma} |y|^{-\beta}$, $(\alpha + \gamma)/n = 1/p' + 1/q - \beta/n \geq 0$. Hence by case (I) T maps L^{pu} boundedly into L^{qv} .

(III) $\beta < 0$, now

$$(26) \quad 1/p' \geq \alpha/n > (\alpha + \beta)/n, \quad 1/q \geq 0.$$

If (a) then $K(x, y) \leq C |x|^{-\alpha} |x - y|^{-\gamma - \beta}$, $(\gamma + \beta)/n = 1/p' + 1/q - \alpha/n \geq 0$, if (b) or (c) then $K(x, y) \leq C |x|^{-\alpha - \beta} |x - y|^{-\gamma}$. By (26) and by case (I) the assertion follows.

REMARKS. In those cases when L^{qv} is a normed space (III) can be obtained from (II) by passing to the adjoint. Suppose $\Omega, \Omega_0, \Omega_1 \geq 0$ are homogeneous of degree 0. Since $\Omega \in L^{n/\alpha}(S^{n-1})$ implies $\Omega | \cdot |^{-\alpha} \in L^{n/\alpha, \infty}$ (by Lemma 3) it is easy to see that if instead of $| \cdot |^\alpha, | \cdot |^{-\beta}$ the more general weight functions $| \cdot |^\alpha \Omega_0, | \cdot |^{-\beta} \Omega_1$ are used, then

$$\|[(\Omega | \cdot |^{-\gamma} * f) \Omega_1 | \cdot |^{-\beta}]\|_{qv} \leq C \|f \Omega_0 | \cdot |^\alpha\|_{pu}$$

provided in addition $\Omega_0^{-1} \in L^{n/(\alpha - \alpha^- - \beta^-)}(S^{n-1})$, $\Omega_1 \in L^{n/(\beta - \alpha^- - \beta^-)}(S^{n-1})$,

$$\Omega \in L^{n/(\gamma - \alpha^- - \beta^-)}(S^{n-1}).$$

Proposition 4 raises the question whether the conditions $p \leq q$ (if $p=u$, $q=v$), $\alpha+\beta \geq 0$ are necessary.

PROPOSITION 5. Suppose

$$Tf(x) = \int \Omega(x-y)|x-y|^{-\gamma} f(y) dy, \quad \omega_0(x) = \Omega_0(x)|x|^\alpha, \\ \omega_1(x) = \Omega_1(x)|x|^{-\beta}$$

where Ω , $\Omega_1 \geq 0$, Ω , Ω_0 , Ω_1 are positively homogeneous of degree 0 and none is 0 a.e. If $\|f\|_{p, \omega_0} = \|f\omega_0\|_p$ then in order for

$$(27) \quad \|Tf\|_{q, \omega_1} \leq C\|f\|_{p, \omega_0}$$

to hold it is necessary that $p \leq q$.

Proof. Define the continuous linear maps $A: L^p(R_+^n, L^p(S^{n-1})) \rightarrow L_{\omega_0}^p$ where the multiplicative group of positive real numbers R_+^n is provided with Haar measure dt/t , by $(Ag)(x) = g(|x|, x')|x|^{-\alpha-n/p}\Omega_0^{-1}(x)$, ($x' = |x|^{-1}x$) and

$$B: L_{\omega_1}^q \rightarrow L^q(R_+^n, L^q(S^{n-1}))$$

by $(Bg)(t)(\sigma) = g(t\sigma)t^{-\beta+n/q}\Omega_1(\sigma)$ for $\sigma \in S^{n-1}$. Then BTA is continuous, linear and different from 0. Also, if for $\mu > 0$, $g^\mu(t) = g(\mu t)$, then

$$BTAg^\mu = \mu^{\alpha+n/p} BT(Ag)^\mu = \mu^{\alpha+n/p-n+\gamma} B(TAg)^\mu = \mu^{\alpha+n/p-n+\gamma+\beta-n/q} (BTAg)^\mu$$

hence necessarily $1/q = 1/p + (\alpha + \beta + \gamma)/n - 1$ and BTA is invariant under multiplication. Theorem 1.1 of [12] is valid for any noncompact locally compact group instead of R^n , hence if $p < \infty$ then $p \leq q$.

The same argument works for relatively dilation invariant operators, i.e., such that for some real number ρ $T(f^\mu) = \mu^\rho (Tf)^\mu$, in particular, for singular integrals. In the present case of integrals, if $p = \infty$, $q < \infty$, BTA must vanish on those g which satisfy $\lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow 0} g(t) = 0$, hence by Fatou's lemma $BTA = 0$. It follows that $T = 0$ whenever $p > q$.

For simplicity let now $\omega_0(x) = |x|^\alpha$, $\omega_1(x) = |x|^{-\beta}$ and

$$L_{\alpha}^{pq} = \{f: \|f\|_{pq, \alpha} = \|f| \cdot |^\alpha\|_{pq} < \infty\}.$$

PROPOSITION 6. Let T be continuous, linear and translation invariant from L_{α}^p to L_{β}^q . If T is not 0 and $p < \infty$ then $\alpha + \beta \geq 0$.

Proof. Let $A_\varepsilon = \{x: \varepsilon < |x| < \varepsilon^{-1}\}$ then there are $\varphi \in L_{\alpha}^p$, $\varepsilon > 0$ such that $\text{supp } \varphi \subset A_\varepsilon$ and $\|(T\varphi)\chi_{A_\varepsilon}\|_{q, -\beta} \geq \varepsilon$. If e is any unit vector

$$\|\varphi(\cdot - \mu e)\|_{p, \alpha}^p \leq \int |\varphi(x)|^p |x|^{\alpha p} dx \max_{\varepsilon \leq |x| \leq \varepsilon^{-1}} (|x + \mu e|^{\alpha p} |x|^{-\alpha p}) \leq C_{\varepsilon, \alpha}^p \mu^{\alpha p}$$

for $\mu \geq 2\varepsilon^{-1}$, say. Likewise

$$\|T\varphi(\cdot - \mu e)\|_{q, -\beta} \geq \|T\varphi\|_{q, -\beta} \min_{\varepsilon \leq |x| \leq \varepsilon^{-1}} (|x + \mu e|^{-\beta} |x|^{\beta}) \geq D_{\varepsilon, \beta} \mu^{-\beta} \quad \text{for } \mu \geq 2\varepsilon^{-1}$$

where $D_{\varepsilon, \beta} > 0$, hence $\mu^{-\alpha-\beta}$ must remain bounded as $\mu \rightarrow \infty$, thus $\alpha + \beta \geq 0$.

For singular integral operators $1/q = 1/p + (\alpha + \beta)/n$, $\alpha + \beta \geq 0$, $p \leq q$ imply $p = q$, $\alpha = -\beta$.

REMARKS. Proposition 6 holds for $L^{p,q}$ spaces. Specifically $T: L^{p,u}_\alpha \rightarrow L^{q,v}_\beta$ under the remaining hypotheses of Proposition 6, if $u < \infty$, implies $T = 0$ or $\alpha + \beta \geq 0$.

Propositions 4, 5, 6 are easily extended to the case of metrics and kernels with generalized homogeneity in the sense of [20] and integration over subspaces of R^n as follows. Let P be an $n \times n$ real matrix and for $\lambda > 0$ denote the linear transformation of R^n $e^{P \log \lambda}$ by $\pi(\lambda)$. It is assumed that $|\pi(\lambda)x| \leq \lambda|x|$ for $0 < \lambda \leq 1$ and any $x \in R^n$. Consequently the relation $|\pi(r(x)^{-1})x| = 1$ for $x \neq 0$ defines a metric with the property $r(\pi(\lambda)x) = \lambda r(x)$. Assume further that $X = \{x : x_j = 0, n_1 + 1 \leq j \leq n\}$ and $Y = \{x : x_j = 0, 1 \leq j \leq n - n_2\}$ are invariant under P . Let $P_1 = P|_X$, $P_2 = P|_Y$ and let a_j be the trace of P_j for $j = 1, 2$. If

$$(28) \quad \begin{aligned} a_1/p' + a_2/q &= \alpha + \beta + \gamma, & 0 \leq \gamma < \min(a_1, a_2), & \quad \alpha + \beta \geq 0, \\ \alpha^+/a_1 &\leq 1/p', & \beta^+/a_2 &\leq 1/q, & \quad 0 < u \leq v \end{aligned}$$

except if $1/p' = \alpha^+/a_1$ or $1/q = \beta^+/a_2$ when $1 = u = v'$ and if $If(x) = \int_Y r(x-y)^{-\gamma} f(y) dy$ then $\|(If)r^{-\beta}\| [L^{qv}(X)] \leq C \|fr^\alpha\| [L^{pu}(Y)]$.

The proof is as that of Proposition 4 except that in place of the convolution theorem Krée's generalization of Kantorovič's inequality (see [16]) contained in Proposition 1 is used. Note that if $K(x, y) = r(x)^{-\beta} r(x-y)^{-\gamma} r(y)^{-\alpha}$ then

$$Y^\infty X^{a_1/\gamma, \infty} K < \infty, \quad X^\infty Y^{a_2/\gamma, \infty} K < \infty.$$

By means of the preceding remarks it is easy to construct an example, somewhat similar to that of [3] showing that the Marcinkiewicz interpolation theorem does not hold above the main diagonal. Let $K_{\alpha, \gamma, \beta}(x, y) = |x|^{-\beta/a_1} (|x|^{1/a_1} + |y|^{1/a_2})^{-\gamma} |y|^{-\alpha/a_2}$ for $X = Y = R$. Then it is easy to see that for $T_{\alpha\gamma\beta} f(x) = \int K_{\alpha\gamma\beta}(x, y) f(y) dy$, $\|T_{\alpha\gamma\beta} f\|_{qv} \leq C \|f\|_{pu}$ provided (28) holds (if $a_1, a_2 \geq 1$, $|x|^{1/a_1} + |y|^{1/a_2}$ is a metric equivalent to r constructed above for $P = \text{diag}(a_1, a_2)$). Let $(1/p_i, 1/q_i)$ ($i=0, 1$) be such that $1/p_0 < 1/p_1$, $1/q_0 < 1/q_1$, $0 \leq 1/p_i \leq 1$, $0 \leq 1/q_i < \infty$ and

$$(29) \quad 1/p_1 - 1/p_0, 1/q_1 - 1/q_0 < 1.$$

Let the line through $(1/p_i, 1/q_i)$ ($i=0, 1$) have equation $a_1/p' + a_2/q = c \geq 0$. By (29) $a_1(1/p'_0 - 1/p'_1) = a_2(1/q_1 - 1/q_0) < \min(a_1, a_2)$. Let γ, α satisfy $a_1(1/p'_0 - 1/p'_1) \leq \gamma < \min(a_1, a_2)$, $\gamma \leq c$, $a_1/p'_0 - \gamma \leq \alpha \leq a_1/p'_1$, thus $\beta = a_1/p'_0 + a_2/q_0 - \alpha - \gamma \leq a_2/q_0$; also $\alpha + \beta = \gamma - c \geq 0$. Hence $T_{\alpha\gamma\beta}$ is of restricted weak types (p_i, q_i) for $i=0, 1$, but it is not of type (p_i, q_i) for $p_i^{-1} = (1-t)p_0^{-1} + tp_1^{-1}$, $q_i = (1-t)q_0^{-1} + tq_1^{-1}$, $0 \leq t \leq 1$, unless $p_t \leq q_t$ by the argument of Proposition 5.

4. Singular integrals in weighted norms. The following proposition weakens the integrability conditions on Ω given in Theorem 6 of [25]. (The conditions have been weakened still further by recent independent joint results of Muckenhoupt and Wheeden.)

PROPOSITION 7. Suppose Ω is homogeneous of degree 0 and has mean value 0 on S^{n-1} , $K(x) = \Omega(x)|x|^{-n}$. Let

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y)f(y) dy, \quad Tf = \lim_{\varepsilon \rightarrow 0} T_\varepsilon f = \text{p.v. } K * f.$$

Suppose T_ε is bounded in L^p independently of $\varepsilon > 0$ for some p , $1 < p < \infty$. If $-n/p < \alpha < n/p'$ and $\Omega \in L^q(S^{n-1})$ where

$$(30) \quad 1/q \leq 1 - |\alpha|/n$$

then T is bounded in L_α^p . If $\alpha = -n/p$ or n/p' then T is bounded from $L_\alpha^{p_1}$ to $L_\alpha^{p_\infty}$.

Finally if T is of weak type $(1, 1)$, i.e., $\|T_\varepsilon f\|_{1,\infty} \leq C\|f\|_1$, C independent of $\varepsilon > 0$, $\Omega \in L^{q,1}(S^{n-1})$ and (30) holds, then T is bounded from L_α^1 to $L_\alpha^{1,\infty}$.

The proof is based on

LEMMA 4. Let $\{T_j\}$, $j=0, \pm 1, \pm 2, \dots$, be a family of linear operators on measurable functions on R^n satisfying

$$(31) \quad \|T_j f\|_{qs} \leq A\|f\|_{pr}, \quad 1 \leq r \leq p \leq q \leq s \leq \infty,$$

and such that $T_j f(x) = 0$ unless the support of f has a nonempty intersection with $\{y : 2^j \leq |y| < 2^{j+1}\}$ and $2^{j-1} \leq |x| < 2^{j+2}$. If $w \geq 0$ is a "weight" function on R^n satisfying $w(x_1)/w(x_2) \leq B$ for $\frac{1}{2} \leq |x_1|/|x_2| \leq 2$ and $T = \sum_{j=-\infty}^{\infty} T_j$, then

$$\|(Tf)w\|_{qs} \leq CAB^2 \|fw\|_{pr}$$

if $p=r$, $q=s$, then C may be chosen to be 3.

Proof. Let χ_j be the characteristic function of $\{x : 2^j \leq |x| < 2^{j+1}\}$, $f_j = f\chi_j$, then $\|f\|_p = (\sum_{j=-\infty}^{\infty} \|f_j\|_p^p)^{1/p}$, also

$$\|f\|_{p,\infty}^{*p} = \sup t^p \lambda_f(t) = \sup t^p \sum_{j=-\infty}^{\infty} \lambda_{f_j}(t) \leq \sum \|f_j\|_{p,\infty}^{*p}$$

or $\|f\|_{p,\infty}^{*p} \leq (\sum \|f_j\|_{p,\infty}^{*p})^{1/p}$. Hence by the complex method of interpolation, $\|f\|_{ps} \leq C(\sum \|f_j\|_{ps}^p)^{1/p}$, $p \leq s \leq \infty$, and by duality $\|f\|_{pr} \geq C(\sum \|f_j\|_{pr}^p)^{1/p}$ for $1 \leq r \leq p$. (This argument duplicates the proof of Lemma 3 and its corollary and the preceding assertion could have been deduced from them.)

$$\begin{aligned} \|T(f)w\|_{qs}^q &\leq C^q \sum_{j=-\infty}^{\infty} \sup_{2^j \leq |x| < 2^{j+1}} w(x)^q \left\| \sum_{|k-j| \leq 1} T_j(f_k) \right\|_{qs}^q \\ &\leq C^q \sum_{j=-\infty}^{\infty} 3^{q-1} \sup_{2^j \leq |x| < 2^{j+1}} w(x)^q \sum_{|k-j| \leq 1} \|T(f_j)\|_{pr}^q \\ &\leq C^q A^q B^{2q} \sum_{j=-\infty}^{\infty} \|f_j w\|_{pr}^q \\ &\leq (3AB^2C)^q \left(\sum \|f_j w\|_{pr}^p \right)^{q/p} \leq (3AB^2C)^q \|fw\|_{pr}^q. \end{aligned}$$

Proof of Proposition 7. Let $T^2(f)(x) = \text{p.v.} \int_{1/2 < |x|/|y| < 2} K(x-y)f(y) dy$, $T_j^2(f) = T^2(f\chi_j)$. Note that for $2^{j-1} \leq |x| < 2^{j+2}$, $2^j \leq |y| < 2^{j+1}$,

$$\begin{aligned} \{x : |x-y| \leq 2^{j-2}\} &\subset \{x : |x-y| \leq |x|/2\} \subset \{x : \tfrac{1}{2} \leq |y|/|x| \leq 2\} \\ &\subset \{x : |x-y| \leq 3|x|\} \subset \{x : |x-y| < 2^{j+4}\}. \end{aligned}$$

Hence $T_j^2(f)(x) = 0$ unless $2^{j-1} < |x| < 2^{j+2}$ and

$$\begin{aligned} |T_j^2(f)(x)| &\leq \left| \text{p.v.} \int_{|x-y| \leq 2^{j+4}} K(x-y)f(y) dy \right| \\ &\quad + \int_{2^{j-2} < |x-y| < 2^{j+4}} |\Omega(x-y)| |x-y|^{-n} |f(y)| dy \\ &= |(T - T_{2^{j-2}})(f)(x)| + 2^{-n(j-2)} (|\Omega| \varphi_j) * f(x) \quad \text{where } \varphi_j = \sum_{|k-j| \leq 3} \chi_k. \end{aligned}$$

Hence if $\sup_{\varepsilon > 0} \|T_\varepsilon(f)\|_{ps} \leq A\|f\|_{pr}$ then

$$\begin{aligned} \|T_j^2(f)\|_{ps} &\leq 2A\|f\|_{pr} + 2^{-n(j-2)} \|\Omega \varphi_j\|_1 \|f\|_{pr} \\ &\leq C(A + \|\Omega\| [L^1(S^{n-1})]) \|f\|_{pr}. \end{aligned}$$

Thus Lemma 4 takes care of $T^2 = \sum_{j=-\infty}^{\infty} T_j^2$.

Let χ denote the characteristic function of $(0, 1)$. It remains to consider the kernels

$$K_1(x, y) = K(x, y)\chi(2|x|^{-1}|y|), \quad K_3(x, y) = K(x, y)\chi(2|x||y|^{-1}).$$

Let also $T^i f(x) = \int K_i(x, y)f(y) dy$ for $i=1, 3$. If $0 < \alpha < n/p'$ then $|K_1(x, y)| \leq 2^\alpha |\Omega(x-y)| |x-y|^{-n+\alpha} |y|^{-\alpha}$ and so if $1 < p < \infty$ then

$$\begin{aligned} \|T^1 f\|_p &\leq \|(|\Omega| |\cdot|^{-n+\alpha}) * (|\cdot|^{-\alpha} f)\|_p \leq C_{p,\alpha} \|\Omega\|_{n/(n-\alpha)} \| |\cdot|^{-\alpha} f \|_{(1/p+\alpha/n)^{-1}} \\ &\leq C_{p,\alpha} \|\Omega\|_{n/(n-\alpha)} \|f\|_p. \end{aligned}$$

If $\alpha = n/p' > 0$ then

$$\|(|\Omega| |\cdot|^{-n/p}) * (|\cdot|^{-n/p'} f)\|_{p\infty} \leq C_p \|\Omega\|_p \| |\cdot|^{-n/p'} f \|_1 \leq C_p \|\Omega\|_p \|f\|_{p1}.$$

Moreover for $0 \leq \alpha < n$,

$$\begin{aligned} \|T^1 f\|_\infty &\leq C_\alpha \sup_x \|\Omega(x-\cdot) |x-\cdot|^{-n} |x|^\alpha \chi(2|x|^{-1}|y|)\|_{n/(n-\alpha), 1} \| |\cdot|^{-\alpha} f \|_{n/\alpha, \infty} \\ &\leq C_\alpha \|\Omega\|_{n/(n-\alpha), 1} \|f\|_\infty. \end{aligned}$$

(Hence for $-n < \alpha \leq 0$, $\|T^3 f\|_1 \leq C_\alpha \|\Omega\|_{n/(n-|\alpha|), 1} \|f\|_1$.)

In case $\alpha \leq 0$, $|K_1(x, y)| \leq 2^\alpha |\Omega(x-y)| |x-y|^{-n} \chi(2|x|^{-1}|y|)$ hence $\|K(x, \cdot)\|_1 \leq C_\alpha \|\Omega\|_1$. Thus if $\alpha = 0$ by (9) of Lemma 1, $\|T^1 f\|_1 \leq C \|\Omega\|_1 \|f\|_1$. Also

$$|K_1(x, y)| \leq (3/2)^{|\alpha|} |\Omega(x-y)| |x-y|^{-n-|\alpha|} |y|^{|\alpha|} \chi(|x-y|^{-1}|y|)$$

hence for $\alpha < 0$, $\|K_1(\cdot, y)\|_1 \leq C|\alpha|^{-1} \|\Omega\|_1$. Thus

$$X^\infty Y^1 K_1 + Y^\infty X^1 K_1 \leq C|\alpha|^{-1} \|\Omega\|_1,$$

hence by Kantorovič's inequality (2), $\|T^1 f\|_p \leq C_\alpha \|\Omega\|_1 \|f\|_p$ for $1 \leq p \leq \infty$. By duality if $-n/p < \alpha < 0$ then $\|T^3 f\|_p \leq C_{p,\alpha} \|\Omega\|_{n/(n+\alpha)} \|f\|_p$, and if $\alpha = -n/p$ then $\|T^3 f\|_{p,\infty} \leq C_p \|\Omega\|_{p'} \|f\|_{p,1}$. If $1 \leq p \leq \infty$, $\alpha > 0$, then $\|T^3 f\|_p \leq C_\alpha \|f\|_p$.

Thus the proposition follows by collection of the preceding estimates. Note also that since $1 < (|x|/|y|)^\varepsilon + (|x|/|y|)^{-\varepsilon}$ for $\varepsilon \in R$, $\|T^i f\|_p \leq C_{p,\alpha}(\Omega) \|f\|_p$ holds also for $1 < p < \infty$, $\alpha = 0$, $q > 1$.

REMARKS. If instead of the weight $|x|^\alpha$ a weight function $\omega(|x|)$ is used satisfying the conditions: $\omega(\tau)\tau^{-\alpha_1}$ is almost decreasing for some $\alpha_1 < n/p'$ (i.e., $\tau \leq \tau'$ implies $\omega(\tau')\tau'^{-\alpha_1} \leq C\omega(\tau)\tau^{-\alpha_1}$) while $\omega(\tau)\tau^{\alpha_3}$ is almost increasing for some $\alpha_3 < n/p$, it follows similarly that if

$$1/q \leq 1 - n^{-1} \max(|\alpha_1|, |\alpha_3|).$$

Then T is a bounded operator on L_ω^p .

EXAMPLE. If $\omega(\tau) = (\tau + a)^\alpha$, $a \geq 0$, $-n/p < \alpha < n/p'$, condition (30) is still sufficient.

It is, of course, well known that under the other assumptions T_ε is uniformly bounded in L^p , $1 < p < \infty$. Proposition 7 as stated, however, immediately extends to kernels and metrics with generalized homogeneity in the sense of [20].

The proof of boundedness in L_α^p of singular integral operators with kernel $\Omega(x)|x|^{-n}$, $\Omega \in L^q(S^{n-1})$ in [25], is based on the boundedness of the integral operator associated with the kernel

$$(32) \quad H(x, y) = |\Omega(x-y)| |x-y|^{-n} (1 - |x|^\alpha |y|^{-\alpha}).$$

This kernel may be of independent interest. Proposition 8 shows that the condition $q > 2(n-1)$ can be replaced by $q > n$ or rather $\Omega \in L^{n,1}(S^{n-1})$.

LEMMA 5. Let $S_1 = \{x : 1/2 \leq |x| < 2\}$, $S_2 = \{x : 1/4 \leq |x| < 4\}$, $f^\lambda(x) = f(\lambda x)$, T linear, such that $\text{supp}(f) \subset S_1$ implies $\text{supp}(Tf) \subset S_2$, $T(f^\lambda) = \lambda^{n/r'}(Tf)^\lambda$ and, for f supported in S_1 , $\|Tf\|_{qv} \leq A \|f\|_{pu}$ where $1 \leq r \leq \infty$, $1/q = 1/p - 1/r'$, $u \leq p$, $v \geq q$. Then $\|Tf\|_{qv} \leq CA \|f\|_{pu}$ for any $f \in L^{pu}$ (if $u = p$, $v = q$ then C can be taken to be 3).

This follows from Lemma 4. Put $T_j f = T(f\chi_j)$.

Lemma 5 and Kantorovič's inequality imply

LEMMA 6. Let $X = Y = R^n$, $K(x, y)$ be homogeneous of degree $-n$ and vanish unless $\frac{1}{2} \leq |y|/|x| \leq 2$. Suppose also

$$\max \left(\sup_{1/4 \leq |x| < 4} \int_{1/2 \leq |y| < 2} |K(x, y)| dy, \sup_{1/4 \leq |y| < 4} \int_{1/2 \leq |x| < 2} |K(x, y)| dx \right) = M < \infty.$$

Then $\|Tf\|_p \leq 3M \|f\|_p$ for $1 \leq p \leq \infty$.

Now returning to the kernel H of (32),

$$\begin{aligned}
 & \sup_{1/4 \leq |x| < 4} \int_{1/2 \leq |y| < 2} |1 - |x|^\alpha |y|^{-\alpha}| |\Omega(x-y)| |x-y|^{-n} dy \\
 & \leq 2^{|\alpha|} \sup_{1/4 \leq |x| \leq 4} \int_{1/2 \leq |y| \leq 2} \frac{||y|^\alpha - |x|^\alpha|}{||y| - |x||} \cdot \frac{|\Omega(x-y)|}{|x-y|^{n-1}} dy \\
 & \leq C_\alpha \sup_{|x| \leq 4} \int_{|y| \leq 2} |\Omega(x-y)| |x-y|^{-n+1} dy \\
 & \leq C_\alpha \int_{|y| \leq 6} |\Omega(y)| |y|^{-n+1} dy \\
 & \leq C_{n,\alpha} \|\Omega \chi(|\cdot|/6)\|_{n,1} \|\cdot\|^{-n+1} \chi(|\cdot|/6)\|_{n/(n-1),\infty} \\
 & \leq C_{n,\alpha} \|\Omega\|_{n,1}.
 \end{aligned}$$

Consequently there holds the following

PROPOSITION 8. *Suppose Ω is homogeneous of degree 0, $\Omega \in L^{n+1}(S^{n-1}) \cap L^q(S^{n-1})$, and q satisfies (30), then for H defined by (32),*

$$\left\| \int H(\cdot, y) f(y) dy \right\|_p \leq C_{p,q}(\Omega) \|f\|_p.$$

(There are analogous weaker conclusions if $p=1$, $\alpha = -n/p$ or $\alpha = n/p'$.)

In view of what was said about the kernels K_1, K_3 in the proof of Proposition 7 it may only be required to observe that, by the remark at the end of the proof of Proposition 7, $|\Omega(x-y)| |x|^{-n} \chi(2|y| |x|^{-1})$ and $|\Omega(x-y)| |y|^{-n} \chi(2|x| |y|^{-1})$, that is, K_1, K_3 with $\alpha=0$ give rise to integral operators of type (p, p) if $1 < p < \infty$.

5. On a generalization of Schur's inequality. It may be worth pointing out that the proof of Strichartz's generalization of Schur's inequality (§1 of [25]) is implicitly based on Young's inequality for convolutions on the multiplicative group of positive real numbers R_+^* and two general simple lemmas.

Young's inequality for convolutions on R_+^* is equivalent to

LEMMA 7. *If the function K on $R_+ \times R_+$ is homogeneous of degree $-1/r$, and satisfies*

$$\int_0^\infty |K(1, y)|^r y^{r/p'-1} dy = M^r < \infty$$

and the linear operator T is defined by $Tf(x) = \int K(x, y) f(y) dy$, then $\|Tf\|_q \leq M \|f\|_p$ (norms are taken with respect to Lebesgue measure) provided p, q, r satisfy $1 \leq p, q, r \leq \infty$, $1/q = 1/p + 1/r - 1$.

In fact Young's inequality applied to $\varphi * \psi(x) = \int_0^\infty \varphi(xy^{-1}) \psi(y) y^{-1} dy$ where $\varphi(x) = f(x)x^{1/p}$, $\psi(x) = g(x)x^{1/r}$ says

$$\left\| \int g(\cdot y^{-1}) (\cdot)^{1/r-1/q} f(y) y^{1/p-1/r-1} dy \right\|_q \leq \|f\|_p \|g\|_r.$$

Now set $g(y) = K(1, y^{-1})y^{-1/p' - 1/r}$ then the left-hand side of the preceding inequality becomes $\|Tf\|_q$ by homogeneity of K while

$$\|g\|_r^r = \int_0^\infty |K(1, y^{-1})|^r y^{-r/p' - 1} dy = \int_0^\infty |K(1, y)|^r y^{r/p' - 1} dy.$$

LEMMA 8. Let $\|K\| [L^p, L^q]$ denote the (quasi-) norm of the operator T , $(Tf)(x) = \int K(x, y)f(y) dy$ from $L^p(Y)$ to $L^q(X)$. Suppose $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, then for $q \geq 1$

$$\|K\| [L^p, L^q] \leq \|K\| [L^p(Y_1), L^q(X_1)] \| [L^p(Y_2), L^q(X_2)],$$

where, of course, $\|K\| [L^p(Y_1), L^q(X_1)](x_2, y_2) = \|K((\cdot, x_2), (\cdot, y_2))\| [L^p(Y_1), L^q(X_1)]$.

Proof. By Minkowski's inequality for integrals

$$\begin{aligned} \left\| \iint K(\cdot, \cdot, y_1, y_2) f(y_1, y_2) dy_1 dy_2 \right\|_q \\ \leq \left\| \iint \|K(\cdot, \cdot, \cdot, y_2)\| [L^p(Y_1), L^q(X_1)] \|f(\cdot, y_2)\|_p dy_2 \right\|_q \\ \leq \|K\| [L^p(Y_1), L^q(X_1)] \| [L^p(Y_2), L^q(X_2)] \| f \|_p. \end{aligned}$$

LEMMA 9. Suppose furthermore $X_2 = Y_2 = G$, a compact or locally compact abelian group, $dx_2 = dy_2 = dg$ denotes Haar measure and

$$K(x_1, x_2, y_1, y_2) = L(x_1, y_1, x_2 y_2^{-1}) \geq 0, \quad p = q \geq 1.$$

Then

$$\|K\| [L^p(Y_1 \times G), L^p(X_1 \times G)] = \left\| \int_G L(\cdot, \cdot, g) dg \right\| [L^p, L^p].$$

Proof. For $\varphi_1 \in L^p(Y_1)$, $\psi_1 \in L^{p'}(X_1)$ let

$$l_{\varphi_1, \psi_1}(g) = \int_{X_1} \int_{Y_1} L(x_1, y_1, g) \varphi_1(y_1) \psi_1(x_1) dy_1 dx_1.$$

Since for $h \geq 0$ it is well known that $\sup \{\|h * f\|_p : \|f\|_p \leq 1\} = \int_G h(g) dg$, it follows that

$$\begin{aligned} \left\| \int_G L(\cdot, \cdot, g) dg \right\| [L^p(Y_1), L^p(X_1)] \\ = \sup \left\{ \int_{X_1} \int_{Y_1} L(x_1, y_1, g) \varphi_1(x_1) \psi_1(y_1) dg dx_1 dy_1 : \|\varphi_1\|_p, \|\psi_1\|_{p'} \leq 1, \right. \\ \left. \varphi_1, \psi_1 \geq 0 \right\} \\ = \sup \left\{ \int_G \int_G l_{\varphi_1, \psi_1}(x_2 y_2^{-1}) \varphi_2(y_2) \psi_2(x_2) dy_2 dx_2 : \|\varphi_i\|_p, \|\psi_i\|_{p'} \leq 1, \right. \\ \left. \varphi_i \geq 0, \psi_i \geq 0, i = 1, 2 \right\} \\ \leq \|K\| [L^p(Y_1 \times G), L^p(X_1 \times G)]. \end{aligned}$$

The reverse inequality follows from Lemma 8 with X_1, Y_1, X_2, Y_2 replaced by G, G, X_1, Y_1 respectively.

As in [25] let now X, Y be cones in R^n , $X' = X \cap S^{n-1}$, $Y' = Y \cap S^{n-1}$. Suppose $K(\lambda x, \lambda y) = \lambda^{-n/r} K(x, y)$ for $\lambda > 0$. Write

$$\begin{aligned} r^{n/q} \int_0^\infty \int_{Y'} K(rx', sy') f(sy') s^{n-1} dy' ds \\ = \int_0^\infty \int_{Y'} r^{n/q} s^{n/p} K(rx', sy') [s^{n/p} f(sy')] dy' ds/s. \end{aligned}$$

It is seen that $\|Tf\|_q \leq M\|f\|_p$ is equivalent to

$$\|\tilde{T}g\| [L^q(X' \times R_+^\times)] \leq M\|g\| [L^p(Y' \times R_+^\times)]$$

where \tilde{T} is defined by $K(rx', sy') = r^{n/q} s^{n/p'} K(rx', sy')$, $g(sy') = s^{n/p} f(sy')$ and $X', Y' R_+^\times$ are provided with euclidean surface measure and multiplication invariant measure respectively. Hence by Lemmas 7 and 8

$$(33) \quad \|Tf\|_q \leq \int_0^\infty \|K(\cdot, s \cdot)\| [L^p(Y'), L^q(X')] s^{n/p' - 1} ds \|f\|_p$$

which is a slight simplification and generalization of Corollary 1 of Theorem 1 of [25]. If $K \geq 0$ and $p=q$ then by Lemma 9

$$\left\| \int_0^\infty K(\cdot, s \cdot) s^{n/p' - 1} ds \right\| [L^p(Y'), L^p(X')] = \|K\| [L^p, L^p].$$

The following restricted weak type results are related to (33) and could be used to prove results at the endpoints of the intervals for $\alpha, \beta, 1/p, 1/q$ in the weighted norm estimates for fractional or singular integral operators.

LEMMA 10. Let K be defined on $R_+ \times R_+$, homogeneous of degree $-1/r$, $1 \leq p < \infty$, $\|K(1, \cdot)\|_{p', \infty}^* = M$ and T is defined as before then

$$(34) \quad \|Tf\|_{q, \infty}^* \leq pM\|f\|_{p, 1}^*$$

provided $0 \leq 1/q = 1/p + 1/r^{-1}$. For $K \geq 0$, pM cannot be replaced by any smaller constant.

This might be considered as a supplementary remark to Lemma 3 of [6] and can be proved as follows.

$$\begin{aligned} \int_0^\infty |K(s, t)| |f(t)| dt &= \int_0^\infty K(1, t/s) s^{-1/r} f(t) dt \leq s^{-1/r} M \int_0^\infty (t/s)^{-1/p'} f^*(t) dt \\ &= s^{-1/r + 1/p'} M \int_0^\infty f^*(t) t^{1/p} dt/t = s^{-1/q} pM\|f\|_{p, 1}^*. \end{aligned}$$

Also for $f=f^*$, $K(1, t) = Mt^{-1/p'}$ equality holds throughout.

PROPOSITION 9. Let $K(x, y)$ be positively homogeneous of degree $-n/r$, $p \geq 1$, $q > 1$, $1/q + 1/p' = 1/r$ and $\|K'\| [L^p(Y'), L^q(X')] = A$ where

$$K'(x', y') = \|K(x', (\cdot)^{1/n} y')\| [L^{p'\infty}(R_+)].$$

Then T is of restricted weak type (p, q) and in fact

$$(35) \quad \|Tf\|_{q\infty}^* \leq A p n^{-1/p' - 1/q} \|f\|_{p1}^*.$$

Proof. Let $\varphi(sy') = f(s^{1/n} y')$, $(\tilde{T}\varphi)(rx') = (Tf)(r^{1/n} x')$. Then

$$(36) \quad \|\varphi\| [L^{p1}(Y' \times R_+)] = n^{1/p} \|f\|_{p1}^*, \quad \|\tilde{T}\varphi\| [L^{q\infty}(X' \times R_+)] = n^{1/q} \|Tf\|_{q\infty}^*$$

where R_+ is provided with Lebesgue measure. For $f(s^{1/n} y')$ on $(R_+, n^{-1} ds)$ is equimeasurable with $f(sy')$ on $(R_+, s^{n-1} ds)$. Also

$$(\tilde{T}\varphi)(rx') = n^{-1} \int_{s^{n-1} \times R_+} K(r^{1/n} x', s^{1/n} y') \varphi(sy') dy' ds.$$

Hence by Lemma 3 and its corollary

$$\begin{aligned} \|\tilde{T}\varphi\| [L^{q\infty}(S^{n-1} \times R_+)] &\leq \|\tilde{T}\varphi\| [L^{q\infty}(R_+)] \| [L^q(S^{n-1})] \\ &\leq n^{-1} p \left\| \int \|K(\cdot, (\cdot)^{1/n} y')\| [L^{p'\infty}(R_+)] \|\varphi(\cdot, y')\| [L^{p1}(R_+)] dy' \right\| [L^q(S^{n-1})] \\ &\leq n^{-1} p A \|\varphi\| [L^{p1}(R_+)] \| [L^p(S^{n-1})] \\ &\leq n^{-1} p A \|\varphi\| [L^{p1}(S^{n-1} \times R_+)] \end{aligned}$$

where $q > 1$ was needed to apply Minkowski's inequality for integrals (see the proof of Lemma 8). This together with (36) implies (35).

The special case $q^* = t^* = p^* = s^*$ of the following lemma is Lemma 1 of [25].

LEMMA 11. Suppose for $T_r f(x) = \int |K(x, y)|^r f(y) dy$, $\|T_r f\|_{q^*, t^*} \leq M^r \|f\|_{p^*, s^*}$ then $\|Tf\|_{qt} \leq B M \|f\|_{ps}$ where B is a constant depending on r, p^*, q^*, s^*, t^* and

$$(37) \quad \begin{aligned} 1 \leq r < \infty, \quad p &= r p^* (1 + (r-1) p^*)^{-1}, \\ s &= r s^* (1 + (r-1) s^*)^{-1}, \quad q = r q^*, \quad t = r t^*. \end{aligned}$$

Proof. Let $f \in L^{ps}$. It can be assumed that $K, f \geq 0$ and f is simple. Then if

$$(38) \quad 1/p = 1/p_0 + 1/p_1, \quad 1/s = 1/s_0 + 1/s_1 \quad (p_i, s_i > 0),$$

then by [13, §3] there exist (simple) $H_0, H_1 \geq 0$ such that $f = H_0 H_1$ and

$$(39) \quad \|H_i\|_{p_i s_i} \leq B (\|f\|_{ps})^{s/s_i} \quad \text{for } i = 0, 1.$$

Hence

$$\begin{aligned} Tf(x) &\leq \left(\int K(x, y)^r H_0(y)^r dy \right)^{1/r} \left(\int H_1(y)^r dy \right)^{1/r}, \\ \|Tf\|_{qt} &\leq \|(T_r(H_0^r))^{1/r}\|_{qt} \|H_1\|_{r'} = \|T_r(H_0^r)\|_{q/r, t/r}^{1/r} \|H_1\|_{r'} \\ &\leq M \|H_0^r\|_{p^* s^*} \|H_1\|_{r'} = M \|H_0\|_{p^* r, s^* r} \|H_1\|_{r'}. \end{aligned}$$

By the second and third relations of (37), $p_0 = p^* r$, $p_1 = r'$, $s_0 = s^* r$, $s_1 = r'$ satisfy (38) and the assertion follows from (39).

REFERENCES

1. A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365. MR **24** #A3479.
2. A. Benedek and R. Panzone, *The spaces L^p , with mixed norm*, Duke Math. J. **28** (1961), 301–324. MR **23** #A3451.
3. C. A. Berenstein, M. Cotlar, N. Kerzman and P. Krée, *Some remarks on the Marcinkiewicz convexity theorem in the upper triangle*, Studia Math. **29** (1967), 79–95. MR **37** #6780.
4. A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc. Sympos. Pure Math., vol. 4, Amer. Math. Soc., Providence, R. I., 1960, pp. 33–49. MR **26** #603.
5. ———, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190. MR **29** #5097.
6. ———, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273–299. MR **34** #3295.
7. ———, *Algebras of singular integral operators*, Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R. I., 1966, pp. 18–55.
8. A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309. MR **18**, 894.
9. ———, *A note on the interpolation of sublinear operations*, Amer. J. Math. **78** (1956), 282–288. MR **18**, 586.
10. R. R. Coifman, *Remarks on weak type inequalities for operators commuting with translations*, Bull. Amer. Math. Soc. **74** (1968), 710–714. MR **37** #6687.
11. E. B. Fabes and N. M. Rivière, *Singular integrals with mixed homogeneity*, Studia Math. **27** (1966), 19–38. MR **35** #683.
12. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140. MR **22** #12389.
13. R. A. Hunt, *On $L(p, q)$ spaces*, Enseignement Math. (2) **12** (1966), 249–276. MR **36** #6921.
14. L. V. Kantorovič, *Integral operators*, Uspehi Mat. Nauk **11** (1956), no. 2 (68), 3–29. (Russian) MR **20** #5432.
15. M. A. Krasnosel'skiĭ, et al., *Integral operators in spaces of summable functions*, "Nauka", Moscow, 1966. (Russian) MR **34** #6568.
16. P. Krée, *Une inégalité sur les transformations intégrales*, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A222–A225. MR **35** #5980.
17. B. Muckenhoupt, *On certain singular integrals*, Pacific J. Math. **10** (1960), 239–261. MR **22** #3948.
18. G. O. Okikiolu, *Bounded linear transformations in L^p space*, J. London Math. Soc. **41** (1966), 407–414. MR **33** #7801.
19. R. O. O'Neil, *Integral transforms*, J. Analyse Math. **21** (1968), 1–276.
20. N. M. Rivière, *On singular integrals*, Bull. Amer. Math. Soc. **75** (1969), 843–847. MR **39** #4711.
21. C. Sadosky, *A note on parabolic fractional and singular integrals*, Studia Math. **26** (1966), 295–302.
22. E. M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. **8** (1957), 250–254. MR **19**, 547.
23. E. M. Stein and G. Weiss, *Fractional integrals on n -dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514. MR **20** #4746.
24. R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), 1031–1060. MR **35** #5927.

- 25. R. S. Strichartz, *L^p estimates for integral transforms*, Trans. Amer. Math. Soc. **136** (1969), 33–50. MR **38** #2638.
- 26. F. Trèves, *Topological vector spaces, distributions, and kernels*, Academic Press, New York, 1967. MR **37** #726.
- 27. A. Zygmund, *Trigonometric series*. Vol. II, 2nd ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

PRINCETON UNIVERSITY,
PRINCETON, NEW JERSEY 08540